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Two notes on locally conformally flat four- and six-manifolds

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ABSTRACT: Several problems of an article by Gursky are pointed out in this paper. A lemma and aninference are modified, and two noteson locally conformally flat four- and six-manifoldsare given.

Key words: Riemannian manifold, conformally flat metric, six-manifold, note

I. INTRODUCTION

In [1], Gursky gave various results. One of the most important results was that compact, boundless, conformally flat four or six-manifold with positive scalar curvature and positive Euler representation must be conformally equivalent to a sphere or projective space. In the proof of this result, the author gave another proof of Yamabe problem without using the positive mass theorem. This article also included: on a compact, boundless,four -manifold with positive scalar curvature and positive Euler representation, there was a L^2 -pinchingphenomenon with respect to Weyl curvature. In addition, the improvement of Bourguignon's vanishing theorem was given

For the convenience of narration, we will firstly explain some marks.Let $(M^n,\bar{g})$ be a compact, boundless $n(n\geq 3)$ -dimensionalRiemannianmanifold.Let $C=C[\bar{g}]=\{u^{\frac{4}{n-2}}\bar{g}:u>0\}$ be the conformal class of $\bar{g}$ [2−3].

If $g=u^{\frac{4}{n-2}}\bar{g}$, there is a conformal scalar curvature equation

$$R_g=-C_n^{-1}\frac{L_{\bar{g}}u}{u^{N-1}}, \qquad (1)$$

where $N=\frac{2n}{n-2}$, $C_n=\frac{n-2}{4(n-1)}$, $L_{\bar{g}}=\Delta_{\bar{g}}-C_nR_{\bar{g}}$ represent conformal Laplacian operators.Equivalently, we can see the following functional $u\in W^{1,2}(M)\mapsto Q[u]$

$$Q[u]=\frac{\int_M|\nabla_{\bar{g}}u|^2\,dV_{\bar{g}}+C_n\int_M R_{\bar{g}}u^2dV_{\bar{g}}}{(\int_M|u|^N\,dV_{\bar{g}})^{2/N}},$$

The critical point of this functional satisfies Euler equation (1), and $R_g\equiv$ const [4−5] .From Sobolev embedding $W^{1,2}$ Ì L^N, we know that $Q[u]$ has a lower bound, and note that its infimum is Q [6].For $P\leq N$, we can define "sub critical"functional

$$Q_P[u]=\frac{\int_M|\nabla_{\bar{g}}u|^2\,dV_{\bar{g}}+C_n\int_M R_{\bar{g}}u^2dV_{\bar{g}}}{(\int_M|u|^P\,dV_{\bar{g}})^{2/P}}.$$

For every P, there is a positive function $u_P\in C^\infty(M)$ such that

$$Q_P[u_P]=Q_P=\inf_{\substack{u\in W^{1,2}\\ u\not\equiv 0}}Q_P[u].$$

If u_P is normalized such that $\|u_P\|_P=1$, then u_P satisfies the Euler-Lagrange equation

$$L_{\bar{g}}u_P=-Q_Pu_P^{p-1}. \qquad (2)$$

Take a little column P_k □ N, and note $u_k=u_{P_k}$, $Q_k=Q_{P_k}$. Let R_k, Δ_k and ∇_k represent the scalar curvature, Laplacian operator and covariant differential with respect to metric $g_k=u_k^{\frac{4}{n-2}}\bar{g}$, respectively.

II. PROOF OF TWO NOTES

In this section, we give two notes on locally conformally flat four- and six-manifolds.

Note1. In [1], in order to prove the conclusion of six-manifold in Theorem A, the author gaveLemma 1.5, that is, R_k satisfied

$$\Delta_kR_k=C_n(N-P_k)R_k^2-C_n(N-P_k)R_{\bar{g}}u_k^{2-N}R_k-\left(\frac{P_k-N-1}{N-P_k}\right)|\nabla_kR_k|^2R_k^{-1}.$$

(3)

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The author mistakenly wrote the molecule $P_k - N - 1$ in the third item on the right of (3) as $P_k - N + 1$. Next we give the proof.

Proof: In the local coordination, we note $g_{ij} := (g_k)_{ij}, g^{ij} := (g_k)^{ij}$. It is known from (1.5) of [1]

$$R_k = C_n^{-1} Q_k u_k^{P_k - N} \geq 0.$$

The left side of (3) is

$$\begin{aligned}
\Delta_k R_k &= (\nabla_i \nabla_j R_k) g^{ij} \\
&= \nabla_i [C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} \nabla_j u_k] g^{ij} \\
&= C_n^{-1} Q_k (P_k - N)(P_k - N - 1) u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&\quad + C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} \Delta_k u_k \\
&= C_n^{-1} Q_k (P_k - N)(P_k - N - 1) u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&\quad + C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} \cdot u_k^{2-N} \Delta_{\bar{g}} u_k.
\end{aligned}$$

The right side of (3) is

$$\begin{aligned}
& C_n (N - P_k) R_k^2 - C_n (N - P_k) R_{\bar{g}} u_k^{2-N} R_k - \left(\tfrac{P_k - N - 1}{N - P_k}\right) |\nabla_k R_k|^2 R_k^{-1} \\
&= C_n (N - P_k) R_k^2 - C_n (N - P_k) u_k^{2-N} R_k (R_k u_k^{N-2} + \tfrac{C_n^{-1} \Delta_{\bar{g}} u_k}{u_k}) \\
&\quad + (P_k - N - 1)(P_k - N) C_n^{-1} Q_k u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&= -C_n^{-1} Q_k^2 (P_k - N) u_k^{2(P_k - N)} - Q_k (N - P_k) u_k^{P_k - 2N + 2} (R_k u_k^{N-2} + \tfrac{C_n^{-1} \Delta_{\bar{g}} u_k}{u_k}) \\
&\quad + (P_k - N - 1)(P_k - N) C_n^{-1} Q_k u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&= C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} [-Q_k u_k^{P_k - N + 1} + C_n u_k^{3-N} (R_k u_k^{N-2} + \tfrac{C_n^{-1} \Delta_{\bar{g}} u_k}{u_k})] \\
&\quad + (P_k - N - 1)(P_k - N) C_n^{-1} Q_k u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&= C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} [-Q_k u_k^{P_k - N + 1} + C_n R_k u_k + u_k^{2-N} \Delta_{\bar{g}} u_k] \\
&\quad + (P_k - N - 1)(P_k - N) C_n^{-1} Q_k u_k^{P_k - N - 2} |\nabla_k u_k|^2 \\
&= C_n^{-1} Q_k (P_k - N) u_k^{P_k - N - 1} \cdot u_k^{2-N} \Delta_{\bar{g}} u_k \\
&\quad + C_n^{-1} Q_k (P_k - N)(P_k - N - 1) u_k^{P_k - N - 2} |\nabla_k u_k|^2,
\end{aligned}$$

The first equal sign in the above formula is due to

$$R_{\bar{g}} = R_k u_k^{N-2} + \frac{C_n^{-1} \Delta_{\bar{g}} u_k}{u_k}$$

and

$$|\nabla_k R_k|^2 = C_n^{-2} Q_k^2 (P_k - N)^2 u_k^{2(P_k - N - 1)} |\nabla_k u_k|^2.$$

The last equal sign in the above formula is due to

$$L_{\bar{g}} u_k = -Q_k u_k^{P_k - 1}.$$

To sum up, the equation (3) holds. □

In this way, we know that in the proof of Theorem A in [1], the derivation after lemma 1.5 is incorrect.

Note2. The inference 2.4 of [1] describes that, let $(M^n, \bar{g})$ be a compact, boundlesssix-manifold, and $R_g \geq 0$, if

$$\int_M |W_g|^2 dV_g < 32\pi^2, \quad (4)$$

then there must be $\chi(M) \leq 2$. Where R_g represents the quantitative curvature of metric g, W represents the Wely curvature of manifold M, and $\chi(M)$ represents the Euler representation of manifold M.

Here we need to explain that in order to get the result of the above inference, we must make some modifications to condition (4), that is, change it to

$$\int_M |W_g|^2 dV_g < \varepsilon < 32\pi^2, \quad (5)$$

Otherwise, the conclusion $\chi(M) \leq 2$ in the inference cannot be obtained. Next we give the proof.

Proof: According to corollary2.3 of [1], for a given $\delta > 0$, there is a metric $h = u^2 g$ (u is a positive function) that satisfies

$$2\int_M |E_h|^2 dV_h + 32\pi^2 (\chi(M) - 2) \leq \int_M |W_h|^2 dV_h + \delta,$$

(6)

where E_h represents the traceless Ricci curvature of metric h.

Substitute (5) into (6) to get

$$0 \leq 2\int_M |E_h|^2 dV_h < -32\pi^2 (\chi(M) - 2 - \frac{\varepsilon}{32\pi^2}) + \delta,$$

From the arbitrariness of $\delta > 0$

$$\chi(M) \leq 2 + \frac{\varepsilon}{32\pi^2} < 3.$$

we have

$$\chi(M) \leq 2. \square$$

From the above proof, we can see that when the condition is (4), the conclusion is $\chi(M) \leq 3$.

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